Advances in Reachability Analysis with Applications to Safety Verification of Vehicle Control Systems

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System is safe, if no trajectory enters the unsafe set.
Safety Verification Using Reachable Sets

- System is safe, if no trajectory enters the unsafe set.
- Overapproximated system is safe $\rightarrow$ real system is safe.
Main Innovations

Consideration of Time-Varying Parameters for Linear Systems

There is much work for linear time invariant (LTI) Systems; a wrapping-free algorithm exists [A. Girard, C. Le Guernic, O. Maler; HSCC 2006].

Here: The system matrix is uncertain and time-varying.

Novel Linearization Approach for Nonlinear Systems

Before: The linearization error is considered by an additional uncertain input.

Here: The linearization error is considered by adding parameter uncertainties.

Continuization of Hybrid Systems

Before: Hybrid dynamics requires intersection of reachable sets with guard sets.

Here: The intersection can be eliminated by temporarily enlarging the set of uncertain parameters.
Considered Class of Systems

Linear systems with uncertain time varying parameters

$$\dot{x}(t) = A(t)x(t) + u(t),$$

where $A : \mathbb{R}^+ \rightarrow \mathcal{A}$, $u : \mathbb{R}^+ \rightarrow \mathcal{U}$ are piecewise continuous, and $\mathcal{A} \subset \mathbb{R}^{n \times n}$, $\mathcal{U} \subset \mathbb{R}^n$. For reachability analysis, we consider all possible functions $A(t)$ and $u(t)$.

Example:

$$\mathcal{A} = \begin{pmatrix} [-1.05, -0.95] & [-4.05, -3.95] \\ [3.95, 4.05] & [-1.05, -0.95] \end{pmatrix}$$

$$\mathcal{U} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} [-0.05, 0.05]$$
Overview of Reachable Set Computation

1. Compute reachable set $\mathcal{H}(r)$ at time $r$ when there is no input. 
   *Input not yet considered.*

2. Obtain convex hull of initial set $\mathcal{R}(0)$ and $\mathcal{H}(r)$. 
   *Curvature of trajectories not yet considered.*

3. Enlarge reachable set to account for (1) uncertain inputs, (2) curvature of trajectories.

4. Continue with further time intervals $[kr, (k + 1)r], k \in \mathbb{N}$. 

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Reachability Analysis of Linear Time Varying Systems

Althoff, Le Guernic, Krogh (CMU,NYU)
Peano Baker Series

Superposition principle: First, consider only the initial state solution

\[ x(t) = \Phi(A(\tau), t)x_0, \]

where \( \Phi(A(\tau), t) \) is referred to as the Peano Baker Series.

\[ \Phi(A(\tau), t) = I + \int_0^t A(\sigma_1)d\sigma_1 + \int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2) d\sigma_2 d\sigma_1 + \int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2) \int_0^{\sigma_2} A(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 + \ldots \]

*How to compute the set \( \{\Phi(A(\tau), t) | A(\tau) \in \mathcal{A}\} \)?*
Overapproximation of the Peano Baker Series

Time discretization: \( \int_0^t A(\sigma_i) d\sigma_i \approx \sum_{i=1}^k A(l_i \Delta) \Delta, \quad t = k \Delta \) (Riemann integration).

Approximate \( \Phi(A(\tau), t) \) iteratively as

\[
\tilde{\Phi}_1(A(\tau), k, \Delta) = I + \sum_{l_1=1}^k A(l_1 \Delta) \Delta,
\]

\[
\tilde{\Phi}_i(A(\tau), k, \Delta) = \tilde{\Phi}_{i-1}(t, \Delta) + \sum_{l_1=1}^k \cdots \sum_{l_i=1}^{l_2} \left( \prod_{q=1}^i A(l_q \Delta) \right) \Delta^i,
\]

Reminder: \( \Phi(A(\tau), t) = I + \int_0^t A(\sigma_1) d\sigma_1 + \int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2) d\sigma_2 d\sigma_1 + \ldots \)
Reachability Analysis of Linear Time Varying Systems

Overapproximation of the Peano Baker Series

1. Time discretization: \( \int_{0}^{t} A(\sigma_i)d\sigma_i \approx \sum_{i=1}^{k} A(l_i \Delta) \Delta, \ t = k \Delta \) (Riemann integration).

2. Replace concrete matrices by sets of matrices.

Approximate \( \Phi(A(\tau), t) \) iteratively as

\[
\tilde{\Phi}_1(A(\tau), k, \Delta) = I + \sum_{l_1=1}^{k} A(l_1 \Delta) \Delta, \\
\in \bigoplus_{l_1=1}^{k} A \Delta
\]

\[
\tilde{\Phi}_i(A(\tau), k, \Delta) = \tilde{\Phi}_{i-1}(t, \Delta) + \sum_{l_i=1}^{k} \cdots \sum_{l_1=1}^{l_2} \left( \prod_{q=1}^{i} A(l_q \Delta) \right) \Delta^i, \\
\in \bigoplus_{l_1=1}^{k} \cdots \bigoplus_{l_1=1}^{l_2} A \Delta^i
\]

where \( \bigoplus \) represents the Minkowski addition: \( A \bigoplus B = \{ A + B \mid A \in A, B \in B \} \).
Overapproximation of the Peano Baker Series

1. Time discretization: \( \int_0^t A(\sigma_i) d\sigma_i \approx \sum_{l_i=1}^k A(l_i \Delta) \Delta, \ t = k \Delta \) (Riemann integration).

2. Replace concrete matrices by sets of matrices.

3. Apply distributivity of convex matrix sets: \( aA \oplus bA = (a + b)A \)

Approximate \( \Phi(A(\tau), t) \) iteratively as

\[
\tilde{\Phi}_1(A(\tau), k, \Delta) \in I \oplus \bigoplus_{l_1=1}^k A \Delta, \\
\subseteq \text{CH}(A) t
\]

\[
\tilde{\Phi}_i(A(\tau), k, \Delta) \in \tilde{\Phi}_{i-1}(t, \Delta) \oplus \bigoplus_{l_1=1}^k \bigoplus_{l_2=1}^{l_i} A^i \Delta^i, \\
\subseteq \frac{1}{i} \text{CH}(A^i) t^i =: \overline{M}_i(t)
\]

where \( \text{CH}() \) is the convex hull operator, which ensures that the distributivity law can be applied.
Overapproximation of the State Transition Matrix

The expressions \( \overline{M}_i(t) \) are independent of \( \Delta \). For \( \lim_{\Delta \to 0} \) we have that

Overapproximation of the state transition matrix

\[
\Phi(A(\tau), t) \in \bigoplus_{i=0}^{\infty} \overline{M}_i(t), \quad \overline{M}_i(t) = \frac{t^i}{i!} \text{CH}(A^i).
\]

Overapproximation of the state transition matrix: time invariant case

\[
\Phi(A, t) \in \left\{ \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \bigg| A \in \mathcal{A} \right\}.
\]
Overapproximation of the State Transition Matrix

The expressions $\overline{M}_i(t)$ are independent of $\Delta$. For $\lim_{\Delta \to 0}$ we have that

Overapproximation of the state transition matrix

$$\Phi(A(\tau), t) \in \bigoplus_{i=0}^{\infty} \overline{M}_i(t), \quad \overline{M}_i(t) = \frac{t^i}{i!} \text{CH}(A^i).$$

Overapproximation of the state transition matrix by a finite sum

$$\Phi_i(A(\tau), t) \in \bigoplus_{i=0}^{\eta} \overline{M}_i(t) \oplus [-W(t), W(t)],$$

$W(t)$: remainder bound
Overview of Reachable Set Computation

1. Compute reachable set $\mathcal{H}(r)$ at time $r$ when there is no input. \textit{done}

2. Obtain convex hull of initial set $\mathcal{R}(0)$ and $\mathcal{H}(r)$. \textit{trivial}

3. Enlarge reachable set to account for (1) uncertain inputs (next slide), (2) curvature of trajectories (skipped).

4. Continue with further time intervals $[kr, (k + 1)r], k \in \mathbb{N}$. 

![Diagram showing steps 1 to 4 of reachable set computation.](image-url)
Input Solution

Removing the input

The differential equation $\dot{x}(t) = A(t)x(t) + u(t)$ can be rewritten as

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ 1 \end{pmatrix} = \begin{pmatrix} A(t) & u(t) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ 1 \end{pmatrix}$$

\[ \text{... analogous proofs ...} \]

Reachable set due to the input

$$\mathcal{P}(t) = \bigoplus_{i=0}^{\eta} \left( \frac{t^{i+1}}{(i+1)!} \text{CH}(A^iU) \right) \oplus \frac{t}{\eta + 2} \left[ -W(t), W(t) \right] |U|.$$
Typical Types of Sets for Reachability Analysis

Polytopes: Convex hull of vertices
\[
\left\{ \sum_{i=1}^{r_A} \alpha_i \mathbf{v}^{(i)} \mid \mathbf{v}^{(i)} \in \mathbb{R}^n, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}
\]

Zonotopes: Minkowski sum of line segments
\[
\{ \mathbf{g}^{(0)} + \sum_{i=1}^{k_A} p_i \mathbf{g}^{(i)} \mid \mathbf{g}^{(i)} \in \mathbb{R}^n, p_i \in [-1, 1] \}
\]

Interval Vector
\[
[a, \bar{a}], \quad \forall i : a_i \leq \bar{a}_i, \quad a, \bar{a} \in \mathbb{R}^n.
\]
Typical Types of Sets for Reachability Analysis

Polytopes: Convex hull of vertices

\[ \left\{ \sum_{i=1}^{r_A} \alpha_i v^{(i)} \mid v^{(i)} \in \mathbb{R}^n, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\} \]

Zonotopes: Minkowski sum of line segments

\[ l_i = [-1, 1]g^{(i)} \]

Interval Vector

\[ [\underline{a}, \bar{a}], \quad \forall i : \underline{a}_i \leq \bar{a}_i, \quad \underline{a}, \bar{a} \in \mathbb{R}^n. \]
Considered Matrix Sets for $A$

Analogous definitions for matrix sets:

Matrix Polytopes: Convex hull of matrices

$$\left\{ \sum_{i=1}^{r_A} \alpha_i V(i) \bigg| V(i) \in \mathbb{R}^{n \times n}, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$

Matrix Zonotopes: Minkowski sum of “matrix line segments“ $L_i = [-1, 1]G^{(i)}$

$$\left\{ G^{(0)} + \sum_{i=1}^{\kappa_A} p_i G^{(i)} \bigg| G^{(i)} \in \mathbb{R}^{n \times n}, p_i \in [-1, 1] \right\}$$

Interval Matrix

$$[A, \overline{A}], \quad \forall i, j : A_{ij} \leq \overline{A}_{ij}, \quad A, \overline{A} \in \mathbb{R}^{n \times n}.$$
Reachability Algorithm

Compute $R([0, t_f])$

$\mathcal{H}_0 = \text{CH}(R(0) \cup \overline{\mathcal{M}}(r)R(0)) \oplus \mathcal{F}(r)R(0)$

$\mathcal{P}_0 = \mathcal{P}(r)$

$R_0 = \mathcal{H}_0 \oplus \mathcal{P}_0$

for $k = 1 \ldots t_f/r - 1$ do

$R_k = \overline{\mathcal{M}}(r)R_{k-1} \oplus \mathcal{P}_0$

end for

$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$
Reachability Analysis of Linear Time Varying Systems

Reachability Algorithm

Compute $R([0, t_f])$

$H_0 = CH(R(0) \cup \overline{M}(r)R(0))$

$F = \mathcal{F}(r)R(0)$

$P_0 = \mathcal{P}(r)$

$P_0 = \mathcal{P}(r)$

$R_0 = H_0 \oplus P_0$

for $k = 1 \ldots t_f/r - 1$ do

$R_k = \overline{M}(r)R_{k-1} \oplus P_0$

end for

$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$
Reachability Analysis of Linear Time Varying Systems

Reachability Algorithm

Compute $R([0, t_f])$

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Compute $R([0, t_f])$

$$\mathcal{H}_0 = \text{CH}(R(0) \cup M(r)R(0)) \oplus F(r)R(0)$$

$$\mathcal{P}_0 = \mathcal{P}(r)$$

$$R_0 = \mathcal{H}_0 \oplus \mathcal{P}_0$$

for $k = 1 \ldots t_f/r - 1$ do

$$R_k = \overline{M(r)}R_{k-1} \oplus \mathcal{P}_0$$

end for

$$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$$
Reachability Algorithm

Compute $R([0, t_f])$

$H_0 = CH(R(0) \cup \overline{M}(r)R(0)) \oplus F(r)R(0)$

$P_0 = P(r)$

$R_0 = H_0 \oplus P_0$

for $k = 1 \ldots t_f/r - 1$ do

$R_k = \overline{M}(r)R_{k-1} \oplus P_0$

end for

$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$
Reachability Algorithm

Compute $R([0, t_f])$

\[ H_0 = CH(R(0) \cup \overline{M}(r)R(0)) \oplus F(r)R(0) \]

\[ P_0 = P(r) \]

\[ R_0 = H_0 \oplus P_0 \]

for $k = 1 \ldots t_f / r - 1$ do

\[ R_k = \overline{M}(r)R_{k-1} \oplus P_0 \]

end for

\[ R([0, t_f]) = \bigcup_{k=1}^{t_f / r} R_{k-1} \]
Reachability Algorithm

Compute $R([0, t_f])$

$\mathcal{H}_0 = \text{CH}(R(0) \cup \overline{M}(r)R(0)) \oplus \mathcal{F}(r)R(0)$

$\mathcal{P}_0 = \mathcal{P}(r)$

$R_0 = \mathcal{H}_0 \oplus \mathcal{P}_0$

for $k = 1 \ldots t_f/r - 1$ do

$R_k = \overline{M}(r)R_{k-1} \oplus \mathcal{P}_0$

end for

$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$
Reachability Algorithm

Compute $R([0, t_f])$

$\mathcal{H}_0 = \text{CH}(R(0) \cup \overline{M}(r)R(0)) \oplus F(r)R(0)$

$\mathcal{P}_0 = \mathcal{P}(r)$

$R_0 = \mathcal{H}_0 \oplus \mathcal{P}_0$

for $k = 1 \ldots t_f/r - 1$ do

$R_k = \overline{M}(r)R_{k-1} \oplus \mathcal{P}_0$

end for

$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$
Examples

Computation Times of Random Examples

- Random examples of linear systems for 100 time intervals are computed.
- The random system matrices might be unstable; but does not change computation time.

Table: Computation times in [s].

<table>
<thead>
<tr>
<th>Dimension n</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval matrix</td>
<td>0.10</td>
<td>0.12</td>
<td>0.33</td>
<td>0.82</td>
<td>3.64</td>
</tr>
<tr>
<td>Matrix zonotope ($\kappa = 1$)</td>
<td>0.13</td>
<td>0.17</td>
<td>0.60</td>
<td>2.65</td>
<td>8.72</td>
</tr>
<tr>
<td>Matrix zonotope ($\kappa = 2$)</td>
<td>0.18</td>
<td>0.30</td>
<td>1.13</td>
<td>4.73</td>
<td>18.77</td>
</tr>
<tr>
<td>Matrix zonotope ($\kappa = 4$)</td>
<td>0.34</td>
<td>0.68</td>
<td>2.60</td>
<td>18.07</td>
<td>98.70</td>
</tr>
</tbody>
</table>

$\kappa$: Number of generator matrices.

computed in MATLAB on an i7 Processor (1.6 GHz) and 6GB memory
Considered maneuver: Braking deceleration of $a_x = -0.7g$ ($g$: gravity constant); acceleration due to steering: $a_y \in [-0.4, 0.4]g$.

Verification task: Can the vehicle roll over?

state vector: $x = [\beta, \dot{\Psi}, \Phi, \dot{\Phi}, \Phi_{t,f}, \Phi_{t,r}, v, \int e(t) \, dt]^T$. 
Dynamics of the Closed Loop System

truck dynamics (blue variables are states, red ones are inputs):

\[
mx_7(\dot{x}_1 + x_2) - ms h\dot{x}_4 = Y_\beta x_1 + Y_\psi(x_7)x_2 + Y_\delta \delta
\]

\[
-l_{xz}\dot{x}_4 + l_{zz}\dot{x}_2 = N_\beta x_1 + N_\psi(x_7)x_2 + N_\delta \delta
\]

\[
(l_{xx} + ms h^2)\dot{x}_4 - l_{xz}\dot{x}_2 = ms gh x_3 + ms hx_7(\dot{x}_1 + x_2) - k_f(x_3 - x_5)
\]

\[
-b_f(x_4 - \dot{x}_5) - k_r(x_3 - x_6) - b_r(x_4 - \dot{x}_6)
\]

\[-r(Y_\beta, f x_1 + Y_\psi, f x_2 + Y_\delta \delta) = m_{u,f}(r - h_{u,f})x_7(\dot{x}_1 + x_2) + m_{u,f} gh_{u,f} x_5
\]

\[-k_{t,f} x_5 + k_f(x_3 - x_5) + b_f(x_4 - \dot{x}_5)
\]

\[-r(Y_\beta, r x_1 + Y_\psi, r x_2) = m_{u,r}(r - h_{u,r})x_7(\dot{x}_1 + x_2) - m_{u,r} gh_{u,r} x_6
\]

\[-k_{t,r} x_6 + k_r(x_3 - x_6) + b_r(x_4 - \dot{x}_6)
\]

\[
\dot{x}_7 = a_x
\]

yaw controller:

\[
\delta = k_1 e + k_2 \int e(t) \, dt, \quad e = \dot{\Psi}_d - \dot{\Psi} = \dot{\Psi}_d - x_2.
\]

<table>
<thead>
<tr>
<th>velocity $x_7$</th>
<th>$[10, 20]$ m/s</th>
<th>$[20, 30]$ m/s</th>
<th>$[30, \infty]$ m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>controller gains</td>
<td>$k_1 = 0.4$</td>
<td>$k_1 = 0.5$</td>
<td>$k_1 = 0.6$</td>
</tr>
<tr>
<td></td>
<td>$k_2 = 1.5$</td>
<td>$k_2 = 2$</td>
<td>$k_2 = 2.5$</td>
</tr>
</tbody>
</table>
Classical reachability analysis of hybrid systems

Reachable set computation is continued across discrete transitions using intersections with guard sets → Overapproximations due to intersections, overall complexity is not $O(n^3)$ anymore.

(a) Reachable set of a hybrid system

(b) Overapproximation due to guard intersection
Reachability analysis using continuization

Reachable set is computed under a larger set of parameter uncertainties when intersecting several invariant sets.

\[ P_{total} = P_1 \quad P_{total} = \text{CH}(P_1 \cup P_2) \quad P_{total} = P_2 \]

- Only applicable if there are no jumps.
- Especially suited if the continuous dynamics does not change much.
Examples

Reachable Set of the Truck

- $x_2$
  - $-0.5$
  - $0$
  - $0.5$
- $x_1$
  - $-0.1$
  - $0$
  - $0.1$
- $x_4$
  - $-1$
  - $0$
  - $1$
- $x_3$
  - $-0.4$
  - $-0.2$
  - $0$
  - $0.2$
- $x_6$
  - $-1$
  - $0$
  - $1$
- $x_8$
  - $-0.1$
  - $0$
  - $0.1$
- $x_5$
  - $-0.1$
  - $0$
  - $0.1$
- $x_7$
  - $10$
  - $20$
  - $30$

Unsafe set
Guard set
Motivation for automatic evasion maneuver
Crash is inevitable → vehicle automatically breaks, or steers, or does both. For velocities greater than \( v = \sqrt{8a_{\text{max}}w} \), steering is more effective than braking.

\( a_{\text{max}} \): maximum acceleration, \( w \): width of the vehicle.
Reachable Set of the Evasive Maneuver

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Next Step: Online Verification

Collaborator: Prof. John Dolan (Robotics Institute CMU)
Case Study For Online Verification

Simplifications:

- constant velocity
- reference trajectory consists of arc segments

→ System is linear.

Computation time including collision checks: 0.39 sec on desktop PC (AMD Athlon64 3700+) in MATLAB.
Conclusions

Reachability Analysis:

- Previous methods for the reachability analysis of LTI systems have been extended to uncertain linear time-varying systems.
- Approach scales well with the number of state variables ($O(n^3)$).
- Continuization is promising for hybrid systems with similar continuous dynamics in adjacent locations.
- Result makes it possible to apply an alternative linearization approach → Further work required.

Automotive Applications:

- Cooperative intersection collision avoidance system (CICAS) with Toyota
- Verification of autonomous cars with the Robotics Institute at CMU.